4 Analysis of digital pulses

4.1 Introduction

Information is transmitted as an energy from a source to a destination. This energy can take the form of light waves, radio waves or even sound waves. Any electronic signal can be analyzed either in the time domain or in the frequency domain. All electrical signals, no matter their shape, can be represented by a series of sine or cosine waves.

The standard form of a single frequency signal is:

$$V(t) = V \sin(2\pi f t + \theta)$$

where *v(t)* is the time varying voltage (V), *V* is the peak voltage (V), *f* the frequency (Hz) of the signal and θ its phase (°)

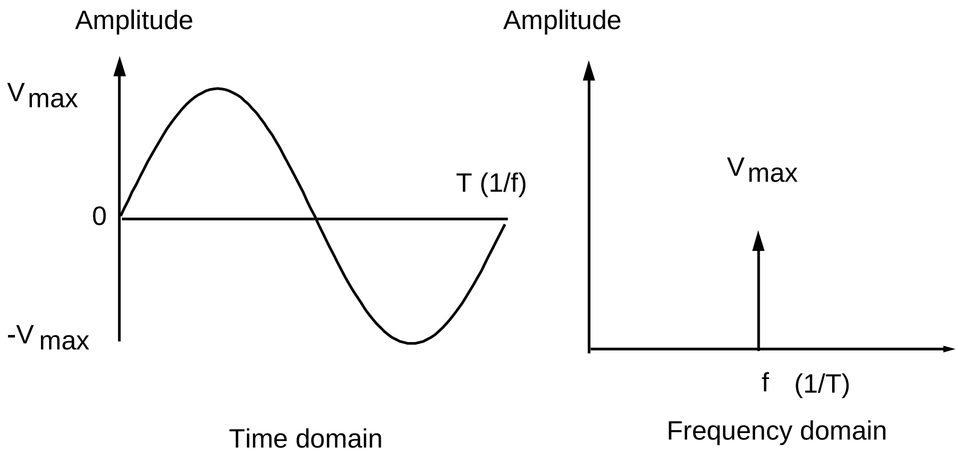

Figure 1 Representation of signal in frequency and time domains

A signal can be represented in the time domain as a varying voltage against time or in the frequency domain as voltage amplitudes against frequency. Figure 1 shows how a sine wave is represented in the time domain and the frequency domain. The signal shown has a period T, the frequency of the signal will be 1/T Hz. This is shown in the frequency domain as a single vertical arrow at that frequency. The amplitude of the arrow represents the amplitude of the signal.

4.2 Repetitive Signals

A repetitive signal is one that repeats after a given time. It can be shown that a repetitive signal is made up of a series of sine and/or cosine waves, called the Fourier series. It can be described by:

$$f(t) = A_0 + A_1 \cos\omega_1 t + A_2 \cos 2\omega_1 t + \ldots + A_N \cos N\omega_1 t$$
$$+ B_1 \sin\omega_1 t + B_2 \sin 2\omega_1 t + \ldots + B_N \sin N\omega_1 t$$

where ω_1 is the fundamental angular frequency ($=2\pi f_1$).

This equation shows that the waveform comprises of an average value (A_0), a series of cosine functions in which each successive term has a frequency that is an integer multiple of the frequency of the first cosine (or sine) in the series. The A and B components can either be found using tables or by using the mathematical formula:

$$A_0 = \frac{1}{T} \int \mathrm{f(t)dt}$$

$$A_N = \frac{2}{T} \int \mathrm{f(t)} \cdot \sin(\mathrm{N}\omega_1 \mathrm{t})\mathrm{dt}$$

$$B_N = \frac{2}{T} \int \mathrm{f(t)} \cdot \sin(\mathrm{N}\omega_1 \mathrm{t})\mathrm{dt}$$

Any periodic waveform has an average, or DC, component and a series of harmonically related sine and cosine waves. A harmonic is an integral multiple of the fundamental frequency. The first harmonic is the fundamental frequency, the second is twice the frequency of the fundamental, the third is three times the multiple, and so on. The fundamental frequency is the lowest frequency in the signal and is thus equal to the inverse of the repetition time. Thus a periodic waveform can be represented by:

f(t) = DC + fundamental + 2nd harmonic + 3rd harmonic + ... + nth harmonic

An example of a repetitive wave is given in Figure 2. It contains a fundamental frequency of amplitude 1 V, a third harmonic amplitude of 0.3 V and fifth harmonic amplitude of 0.2 V. The equation for this wave is:

$$f(t) = \sin(\omega_1 t) + 0.3\sin(3\omega_1 t) + 0.2\sin(5\omega_1 t)$$

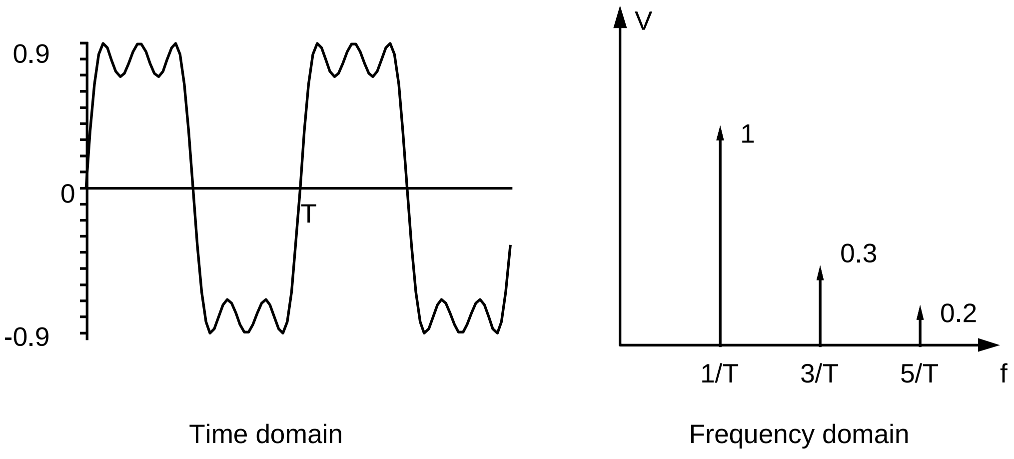

Figure 2 Time and frequency domain representation of a repetitive signal

4.3 Wave symmetry

If a periodic signal is symmetrical about either the vertical or horizontal axis then either the cosine terms or the sine terms become zero.

4.3.1 Even symmetry

When a periodic signal is symmetrical about the vertical axis then it is an even function and the B coefficients in the Fourier equation become zero. Thus the waveform contains only cosine components and a DC level. An example of this type of waveform is given in Figure 3.

With this function f(t) = f(–t), thus the resulting equation will be:

$$f(t) = A_0 + A_1 \cos \omega_1 t + A_2 \cos 2\omega_1 t + \ldots + A_N \cos N\omega_1 t$$

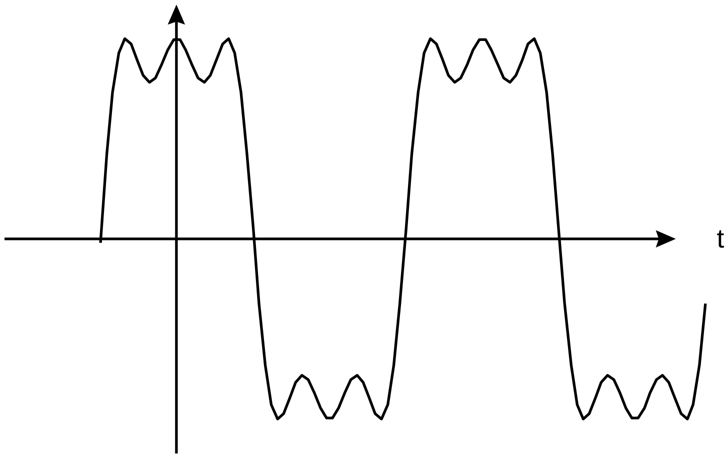

Figure 3 Even symmetry

4.3.2 Odd symmetry

When a periodic signal is symmetrical about the line midway between the vertical and horizontal axis it is an odd function and the A coefficients in the Fourier equation are then zero. Thus the waveform will contain only sine components, with no DC offset. An example of this type of waveform is given in Figure 4.

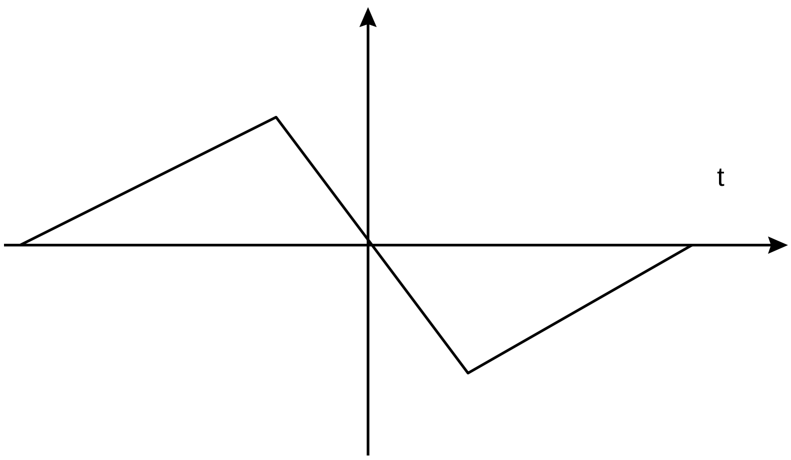

Figure 4 Odd symmetry

With this function f(t) = –f(–t), thus the resulting equation will be:

$$f(t)=B_1 \sin\omega_1 t + B_2 \sin 2\omega_1 t + ... + B_N \sin N\omega_1 t$$

4.3.3 Half-wave symmetry

When the second half cycle of periodic signal is the same as the first half, but is the inverse, then it has half-wave symmetry. The even harmonics in this wave become zero and the waveform will only contain odd harmonics (1st, 3rd, 5th, .., and so on).

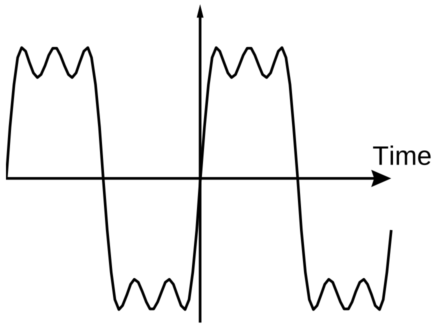

Figure 5 Half-wave symmetry

4.4 Fourier series of a repetitive rectangular waveform

The signal shape of most interest in data communications is the repetitive rectangular pulse, as shown in Figure 6. It is defined by its amplitude and its duty cycle, which is the ratio of the active time of the pulse (τ) to the period of the waveform (T). The duty cycle is thus given by:

$$Duty\ Cycle=\frac{t}{T}$$

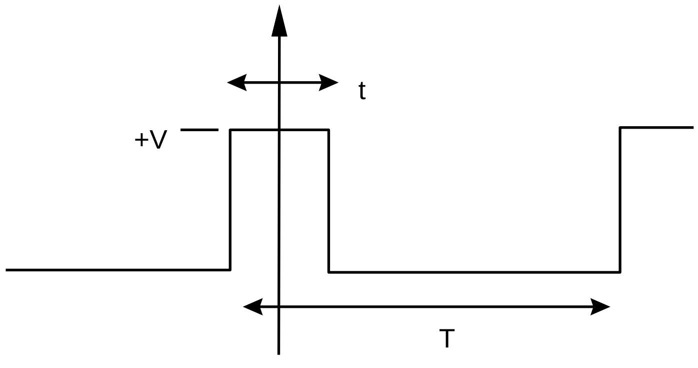

Figure 6 Repetitive pulse waveform

The time-based repetitive pulse waveform is given by:

$$v(t) = \frac{V\tau}{T} + \sum_{n=1}^{n=\infty} V_n \cos(n2\pi f_1 t)$$

the amplitudes of the harmonics is given by:

$$V_n = \frac{2V\tau}{T} . \frac{\sin Nx}{x}$$

where

$$x = \frac{\pi\tau}{T}$$

V_1 is the amplitude of the fundamental, V_2 is the amplitude of the second harmonic, etc. The frequencies contained in the signal will be:

$$f_1 = \frac{1}{T} \text{ Hz}, \ f_2 = \frac{2}{T} \text{ Hz}, \ f_3 = \frac{3}{T} \text{ Hz}, \text{ etc.}$$

The DC component of the signal is thus:

$$V . \frac{\tau}{T}$$

The RMS voltage of a repetitive signal with peak voltage harmonics V_1.. V_n and DC component V_0 is given by the formula:

$$V_{rms} = \sqrt{V_0^2 + \frac{V_1^2}{2} + \frac{V_2^2}{2} + \ldots\ldots \frac{V_n^2}{2}}$$

where V_0 is the DC voltage, V_1 the peak amplitude of first harmonic, and so on. It can be seen that the amplitudes of the harmonics varies as the sin(x)/x function. A typical sin(x)/x function is shown in Figure 7.

Figure 8 gives an example of a repetitive pulse train with a duty cycle of 0.2 and a pulse amplitude of 1 V.

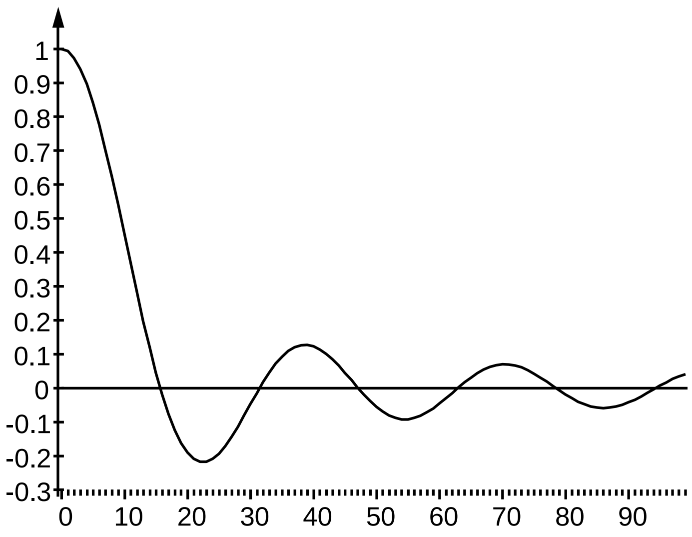

Figure 7 Sin(x)/x function

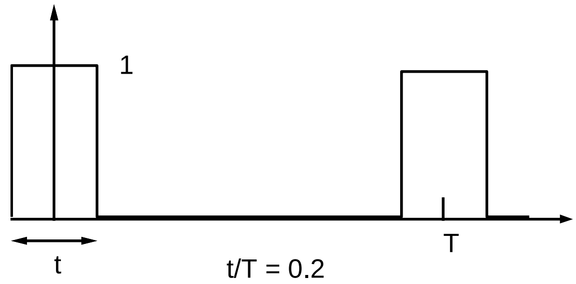

Figure 8 Pulse train with a duty cycle of 0.2

The corresponding Fourier series is given by:

$$v(t) = \frac{Vt}{T} + \sum_{N=1}^{\infty}\left[\frac{2Vt}{T}\cdot\frac{\sin(N\pi\, t/T)}{N\pi\, t/T}\right]\cos(N\omega t)$$

Figure 9 shows the amplitudes of the frequency harmonics.

4.5 Examples

Repetitive pulses of 5 V amplitude, pulse width of 5 μs and repetition time of 25 μs is applied to a communications channel which can be modelled as an ideal low-pass filter with a pass band up to 140 kHz. Figure 10 shows the pulse train.

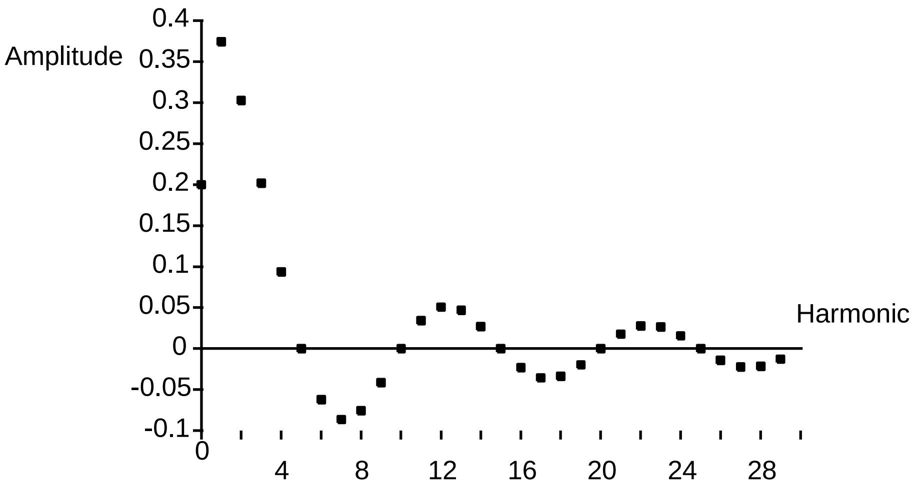

Figure 9 Frequency spectrum (for duty cycle of 0.2)

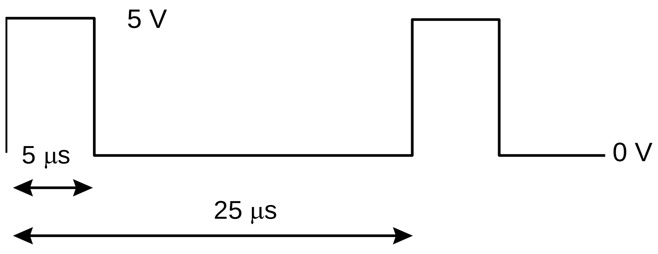

Figure 10 Repetitive pulse train

Determine:

(i) DC voltage offset of the input signal;
(ii) first five harmonic frequencies of the input signal;
(iii) amplitude of the first five harmonics in the input signal.

Also, sketch the time domain response, over a period of 25 µs, of the output signal.

ANSWER
The time response will be:

$$v(t) = \frac{Vt}{T} + \sum_{N=1}^{\infty}\left[\frac{2Vt}{T}.\frac{\sin(N\pi \, {}^{t}\!/\!_{T})}{N\pi \, {}^{t}\!/\!_{T}}\right]\cos(N\omega_1 t)$$

(i) DC offset:

$$V_{DC} = V_{pk}\frac{t}{T} = 5.\frac{5}{25} = 1\ V$$

(ii) First five frequencies:

$$f_1 = \frac{1}{T} = \frac{1}{25\times10^{-6}} = 40\ kHz$$
$$f_2 = 80\ kHz$$
$$f_3 = 120\ kHz$$
$$f_4 = 160\ kHz$$
$$f_5 = 200\ kHz$$

(iii) Amplitude of first five harmonics:

$$V_N = \frac{2Vt}{T}.\frac{\sin(N\pi \, {}^{t}\!/\!_{T})}{N\pi \, {}^{t}\!/\!_{T}}$$

Thus:

$$V_N = \frac{2\times 5\times 5}{25}.\frac{\sin(0.2N\pi)}{0.2N\pi}$$
$$= \frac{3.18}{N}.\sin(0.63N) \quad \text{V}$$

Thus:

N	*f (kHz)*	*V amplitude (Volts)*
1	40	1.87
2	80	1.51
3	120	1.01
4	160	0.47
5	200	0

$$v_i(t)=1+1.87\sin(\omega_1 t)+1.51\sin(2\omega_1 t)+1.01\sin(3\omega_1 t)$$
$$+0.47\sin(4\omega_1 t)+\ldots\ldots$$

assuming filter blocks above 140 kHz, then the output will be:

$$v_o(t)=1+1.87\sin(\omega_1 t)+1.51\sin(2\omega_1 t)+1.01\sin(3\omega_1 t) \quad \text{V}$$

This gives the following table:

ωt (°)	V_0	V_1	V_2	V_3	*Σ*
	1	1.77 cosωt	1.51 cos2ωt	1.01 cos3ωt	
45	1	1.32	0	–0.71	1.61
90	1	0	–1.51	0	–0.51
135	1	–1.32	0	0.71	0.39
180	1	–1.87	1.51	–1.01	–0.37
225	1	–1.32	0	0.71	0.39
270	1	0	–1.51	0	–0.51
315	1	1.32	0	–0.71	1.61
0,360	1	1.87	1.51	1.01	5.39

The pulse output time response can now be plotted for one cycle. Figure 11 shows a rough sketch of the output pulse. The shape of the output would be much smoother if more time points were taken.

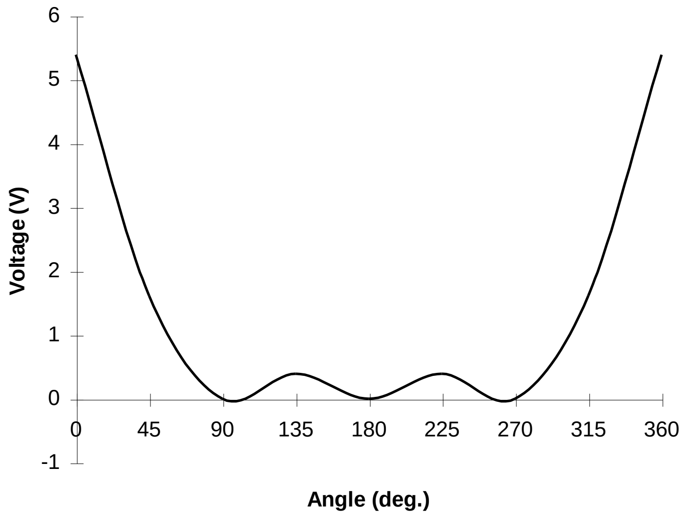

Figure 11 Pulse output

The frequency domain of the output has a DC value of 1 V, a fundamental frequency of 40 kHz, amplitude 1.87 V; a second harmonic at 80 kHz, amplitude 1.51 V; a third harmonic at 120 kHz, amplitude 1.01 V; a forth harmonic at 160 kHz, amplitude 0.47 V and there is no fifth harmonic. A diagram of this is given in Figure 12.

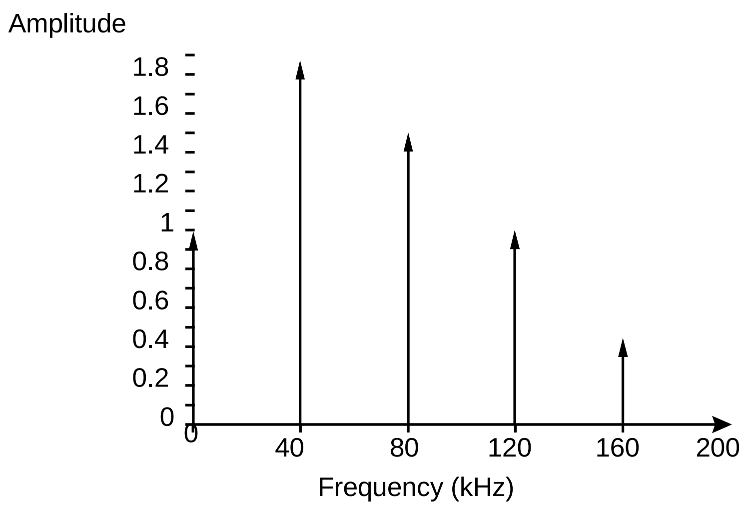

Figure 12 Frequency response of output